

Static electromagnetic fields and charged black holes in general covariant theory of Hořava-Lifshitz gravity

Ahmad Borzou ^{a,*}, Kai Lin ^{a,b,†} and Anzhong Wang ^{a,c,‡}

^a GCAP-CASPER, Physics Department, Baylor University, Waco, TX 76798-7316, USA

^b Institute of Theoretical Physics, Chongqing University, Chongqing 400030, China

^c Institute for Advanced Physics & Mathematics,
Zhejiang University of Technology, Hangzhou 310032, China

(Dated: January 24, 2012)

In this paper, we study electromegnetic static spacetimes in the nonrelativistic general covariant theory of the Hořava-Lifshitz (HL) gravity, proposed recently by Hořava and Melby-Thompson, and present all the electric static solutions, which represent the generalization of the Reissner-Nordström solution found in Einstein's general relativity (GR). The global/local structures of spacetimes in the HL theory in general are different from those given in GR, because the dispersion relations of test particles now contain high-order momentum terms, so the speeds of these particles are unbounded in the ultraviolet (UV). As a result, the conception of light-cones defined in GR becomes invalid and test particles do not follow geodesics. To study black holes in the HL theory, we adopt the geometrical optical approximations, and define a horizon as a (two-closed) surface that is free of spacetime singularities and on which massless test particles are infinitely redshifted. With such a definition, we show that some of our solutions give rise to (charged) black holes, although the radii of their horizons in general depend on the energies of the test particles.

PACS numbers: 04.60.-m; 98.80.Cq; 98.80.-k; 98.80.Bp

I. INTRODUCTION

Recently, Hořava proposed a theory of quantum gravity [1], motivated by the Lifshitz scalar field theory in solid state physics [2]. Due to several remarkable features, the HL theory has attracted a great deal of attention (see for example, [3, 4] and references therein). In this theory, the general covariance is broken down to the foliation-preserving diffeomorphisms $\text{Diff}(M, \mathcal{F})$,

$$\tilde{t} = f(t), \quad \tilde{x}^i = \zeta^i(t, \mathbf{x}), \quad (1.1)$$

because of which, in comparison with GR one more degree of freedom appears in the gravitational sector - the spin-0 graviton. This is potentially dangerous, and needs to decouple in the infrared (IR), in order to be consistent with observations. Whether this is possible or not is still an open question [3]. In particular, Mukohyama studied the spherically symmetric static spacetimes [4], and showed explicitly that the spin-0 graviton indeed decouples after nonlinear effects are taken into account, an analogue of the Vainshtein effect [5], initially found in massive gravity [6]. Similar considerations in cosmology were presented in [7, 8] (See also [9] for a class of exact solutions.), where a fully nonlinear analysis of super-horizon cosmological perturbations was carried out, by adopting the so-called gradient expansion method [10]. It was found that the relativistic limit of the HL theory is continuous, and GR is recovered at least in two

different cases: (a) when only the “dark matter as an integration constant” is present [7]; and (b) when a scalar field and the “dark matter as an integration constant” are present [8].

Another very attractive approach is to eliminate the spin-0 graviton by introducing two auxiliary fields, the $U(1)$ gauge field A and the Newtonian prepotential φ , by extending the $\text{Diff}(M, \mathcal{F})$ symmetry (1.1) to include a local $U(1)$ symmetry [11],

$$U(1) \ltimes \text{Diff}(M, \mathcal{F}). \quad (1.2)$$

Under this extended symmetry, the special status of time remains, so that the anisotropic scaling between space and time,

$$\mathbf{x} \rightarrow b^{-1}\mathbf{x}, \quad t \rightarrow b^{-3}t, \quad (1.3)$$

can still be realized, and the theory is UV complete. Meanwhile, because of the elimination of the spin-0 graviton, its IR behavior can be significantly improved. The elimination of the spin-0 graviton was done initially in the case $\lambda = 1$ [11, 12], but soon generalized to the case with any λ [13–15], where λ denotes a coupling constant that characterizes the deviation of the kinetic part of the action from the corresponding one given in GR. (For the analysis of the Hamiltonian structure of the theory, see [11, 16].)

Under the coordinate transformations (1.1), the lapse function N , the shift vector N_i , the 3-metric g_{ij} , the $U(1)$ gauge field A and the Newtonian prepotential φ transform, respectively, as

$$\begin{aligned} \delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \end{aligned}$$

*Electronic address: ahmad.borzou@baylor.edu

†Electronic address: k.lin@baylor.edu

‡Electronic address: anzhong.wang@baylor.edu

$$\begin{aligned}
\delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}, \\
\delta A &= \zeta^i \partial_i A + \dot{f} A + f \dot{A}, \\
\delta \varphi &= f \dot{\varphi} + \zeta^i \partial_i \varphi,
\end{aligned} \tag{1.4}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to g_{ij} , and $\delta g_{ij} \equiv \tilde{g}_{ij}(t, x^k) - g_{ij}(t, x^k)$, etc. From these expressions one can see that N and N_i play the role of gauge fields of the $\text{Diff}(M, \mathcal{F})$. Therefore, it is natural to assume that N and N_i inherit the same dependence on space and time as the corresponding generators [1],

$$N = N(t), \quad N_i = N_i(t, x), \tag{1.5}$$

which is often referred to as the projectability condition. On the other hand, under the $U(1)$ gauge transformation, the above quantities transform as

$$\begin{aligned}
\delta_\alpha N &= 0, \quad \delta_\alpha N_i = N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0, \\
\delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha,
\end{aligned} \tag{1.6}$$

where $\alpha = \alpha(t, x)$ is the generator of the local $U(1)$ gauge symmetry, and $N^i = g^{ik} N_k$. For the detail, we refer readers to [11, 12].

It should be noted that all the above hold only in the case with projectability condition [1, 3]. However, the elimination of the spin-0 graviton can be also realized in the non-projectability case with the extended symmetry (1.2) [17, 18]. In addition, the number of independent coupling constants can be significantly reduced (from more than 70 [19, 20] to 15), by simply imposing the softly breaking detailed balance condition, while the theory still remains UV complete and has a healthy IR limit. When applying it to cosmology, a remarkable result is obtained: the Friedmann-Robertson-Walker universe is necessarily flat.

In this paper, we study electromagnetic static spacetimes in the Hořava-Melby-Thompson (HMT) setup [11], in which $\lambda = 1$ and the projectability condition (1.5) is adopted. Specifically, after giving a brief introduction to the HMT theory in Sec. II, we consider its coupling to a vector field in Sec. III, while in Sec. IV we study electric static spacetimes, and find all the static solutions. In Sec. V we study the existence of horizons and show that some of these solutions found in Sec. IV have black hole structures in the geometrical optical approximations [21]. It should be noted that because of the breaking of the general covariance, $\tilde{x}^\mu = \zeta^\mu(t, x)$, ($\mu = 0, 1, 2, 3$), the dispersion relations of test particles, including the massless ones, contain generically high-order momentum terms, so the speeds of these particles are unbounded in the ultra-violet (UV), and the causal structures of spacetimes are quite different from that presented in GR. In particular, in the HL theory the conception of light-cones is not valid any longer, and the motions of test particles do not follow geodesics [21]. As a result, the definitions of black holes given in GR [22–25] cannot be applied to the HL gravity without modifications. In Sec. VI we present our main conclusions.

It should be noted that electromagnetic static spacetimes in other versions of the HL theory were studied in [26], while a new mechanism for generation of primordial magnetic seed field in the early universe without the local $U(1)$ symmetry was considered in [27].

II. NONRELATIVISTIC GENERAL COVARIANT THEORY

In this section, we give a very brief introduction to the nonrelativistic general covariant theory of gravity, proposed recently by HMT. For details, we refer readers to [11, 12]. We shall closely follow [12], so that the notations and conventions will be used directly from there without further explanations.

The basic variables are A , φ , N , N_i and g_{ij} , in terms of which the spacetime is given by,

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2.1}$$

The total action takes the form,

$$\begin{aligned}
S &= \zeta^2 \int dt d^3x N \sqrt{g} \left(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A \right. \\
&\quad \left. + \frac{1}{\zeta^2} \mathcal{L}_M \right),
\end{aligned} \tag{2.2}$$

where $g = \det g_{ij}$, and

$$\begin{aligned}
\mathcal{L}_K &= K_{ij} K^{ij} - K^2, \\
\mathcal{L}_\varphi &= \varphi \mathcal{G}^{ij} (2K_{ij} + \nabla_i \nabla_j \varphi), \\
\mathcal{L}_A &= \frac{A}{N} (2\Lambda_g - R).
\end{aligned} \tag{2.3}$$

Here the coupling constant Λ_g , acting like a 3-dimensional cosmological constant, has the dimension of $(\text{length})^{-2}$. K_{ij} is the extrinsic curvature of the hypersurfaces $t = \text{Constant}$, and \mathcal{G}_{ij} is the 3-dimensional “generalized” Einstein tensor, defined, respectively, by

$$\begin{aligned}
K_{ij} &= \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \\
\mathcal{G}_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij},
\end{aligned} \tag{2.4}$$

where the Ricci tensor R_{ij} refers to the three-metric g_{ij} , and $R [= g^{ij} R_{ij}]$ denotes the 3D Ricci scalar. \mathcal{L}_M is the matter Lagrangian and \mathcal{L}_V an arbitrary $\text{Diff}(\Sigma)$ -invariant local scalar functional built out of the spatial metric g_{ij} , its Riemann tensor and spatial covariant derivatives, without the use of time derivatives. In [28], by assuming that the highest order derivatives are six and that the theory respects the parity and time-reflection symmetry, the most general form of \mathcal{L}_V is given by [28] (See also [29]),

$$\mathcal{L}_V = \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij})$$

$$+\frac{1}{\zeta^4} \left(g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_j^i R_k^j R_i^k \right) - \frac{1}{\zeta^4} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk})], \quad (2.5)$$

where the coupling constants g_s ($s = 0, 1, 2, \dots, 8$) are all dimensionless, and

$$\Lambda = \frac{1}{2} \zeta^2 g_0, \quad (2.6)$$

denotes the cosmological constant. The relativistic limit in the IR requires,

$$g_1 = -1, \quad \zeta^2 = \frac{1}{16\pi G}, \quad (2.7)$$

where G denotes the Newtonian constant.

Variation of the total action (2.2) with respect to the lapse function $N(t)$ yields the Hamiltonian constraint,

$$\int d^3x \sqrt{g} (\mathcal{L}_K + \mathcal{L}_V - \varphi \mathcal{G}^{ij} \nabla_i \nabla_j \varphi) = 8\pi G \int d^3x \sqrt{g} J^t, \quad (2.8)$$

where

$$J^t = 2 \frac{\delta(N\mathcal{L}_M)}{\delta N}. \quad (2.9)$$

Variation of the action with respect to the shift vector N_i yields the supermomentum constraint,

$$\nabla_j (\pi^{ij} - \varphi \mathcal{G}^{ij}) = 8\pi G J^i, \quad (2.10)$$

where the supermomentum π^{ij} and matter current J^i are defined as

$$\begin{aligned} \pi^{ij} &\equiv -K^{ij} + K g^{ij}, \\ J^i &\equiv -N \frac{\delta \mathcal{L}_M}{\delta N_i}. \end{aligned} \quad (2.11)$$

Similarly, variations of the action with respect to φ and A yield, respectively,

$$\mathcal{G}^{ij} (K_{ij} + \nabla_i \nabla_j \varphi) = 8\pi G J_\varphi, \quad (2.12)$$

$$R - 2\Lambda_g = 8\pi G J_A, \quad (2.13)$$

where

$$J_\varphi \equiv -\frac{\delta \mathcal{L}_M}{\delta \varphi}, \quad J_A \equiv 2 \frac{\delta(N\mathcal{L}_M)}{\delta A}. \quad (2.14)$$

On the other hand, variation with respect to g_{ij} leads to the dynamical equations,

$$\begin{aligned} \frac{1}{N\sqrt{g}} [\sqrt{g} (\pi^{ij} - \varphi \mathcal{G}^{ij})]_{,t} &= -2 (K^2)^{ij} + 2K K^{ij} \\ &+ \frac{1}{N} \nabla_k [N^k \pi^{ij} - 2\pi^{k(i} N^{j)}] \\ &+ \frac{1}{2} (\mathcal{L}_K + \mathcal{L}_\varphi + \mathcal{L}_A) g^{ij} \\ &+ F^{ij} + F_\varphi^{ij} + F_A^{ij} + 8\pi G \tau^{ij}, \end{aligned} \quad (2.15)$$

where $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$, and

$$\begin{aligned} (K^2)^{ij} &\equiv K^{il} K_l^j, \\ F_A^{ij} &\equiv \frac{1}{N} [A R^{ij} - (\nabla^i \nabla^j - g^{ij} \nabla^2) A], \\ F^{ij} &\equiv -\frac{1}{\sqrt{g}} \frac{\delta(\sqrt{g} \mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^8 g_s \zeta^{n_s} (F_s)^{ij}, \\ F_\varphi^{ij} &\equiv \sum_{n=1}^3 F_{(\varphi, n)}^{ij}, \end{aligned} \quad (2.16)$$

with $n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4)$. The 3-tensors $(F_s)_{ij}$ and $F_{(\varphi, n)}^{ij}$ are given in Appendix A. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g} \mathcal{L}_M)}{\delta g_{ij}}. \quad (2.17)$$

The matter, on the other hand, satisfies the conservation laws,

$$\begin{aligned} \int d^3x \sqrt{g} \left[\dot{g}_{kl} \tau^{kl} - \frac{1}{\sqrt{g}} (\sqrt{g} J^t)_{,t} + \frac{2N_k}{N\sqrt{g}} (\sqrt{g} J^k)_{,t} \right. \\ \left. - 2\dot{\varphi} J_\varphi - \frac{A}{N\sqrt{g}} (\sqrt{g} J_A)_{,t} \right] = 0, \end{aligned} \quad (2.18)$$

$$\begin{aligned} \nabla^k \tau_{ik} - \frac{1}{N\sqrt{g}} (\sqrt{g} J_i)_{,t} - \frac{J^k}{N} (\nabla_k N_i - \nabla_i N_k) \\ - \frac{N_i}{N} \nabla_k J^k + J_\varphi \nabla_i \varphi - \frac{J_A}{2N} \nabla_i A = 0. \end{aligned} \quad (2.19)$$

III. COUPLING OF A VECTOR FIELD

To couple the gravitational sector $(N, N_i, g_{ij}, A, \varphi)$ with a vector field, we borrow the recipe of [13], in which it was shown that for any given matter field, say, ψ_n , that is invariant under $\text{Diff}(M, \mathcal{F})$, its motion is described by the action, $\hat{S}_M(N, N_i, g_{jk}; \psi_n)$. Then, the action,

$$\begin{aligned} S_M(N, N_i, g_{jk}, A, \varphi; \psi_n) &= \hat{S}_M(N, \hat{N}_i, g_{jk}; \psi_n) \\ &+ \int dt d^3x N \sqrt{g} Z(g_{ij}, \psi_n) (A - \mathcal{A}), \end{aligned} \quad (3.1)$$

has the enlarged symmetry (1.2), where $Z(g_{ij}, \psi_n)$ denotes the most general scalar operator of dimension two, $[Z] = 2$, of the enlarged symmetry, and

$$\begin{aligned} \hat{N}_i &\equiv N_i + N \nabla_i \varphi, \\ \mathcal{A} &\equiv -\dot{\varphi} + N^k \nabla_k \varphi + \frac{1}{2} N (\nabla_k \varphi) (\nabla^k \varphi). \end{aligned} \quad (3.2)$$

In [29], the general action of a massive vector field (A_0, A_i) with the $\text{Diff}(M, \mathcal{F})$ symmetry (1.1) was constructed. Applying the above recipe to this vector field,

we obtain the action of a massive vector field that is invariant under the enlarged symmetry (1.2),

$$S_{EM} = \frac{1}{4g_e^2} \int dt d^3x \sqrt{g} \left[\frac{2}{N} g^{ij} (\mathcal{F}_{0i} - \hat{N}^k \mathcal{F}_{ki}) \right. \\ \times (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) + \frac{m_e^2}{N} (A_0 - \hat{N}^i A_i)^2 \\ \left. - N\mathcal{G} + 4g_e^2 \mathcal{K} B_i B^i (A - \mathcal{A}) \right], \quad (3.3)$$

where m_e denotes the mass of the vector field, \mathcal{K} is an arbitrary function of $A^k A_k$, and

$$\mathcal{F}_{0i} = \partial_i A_0 - \partial_t A_i, \quad \mathcal{F}_{ij} = \partial_j A_i - \partial_i A_j, \\ B_i = \frac{1}{2} \frac{\varepsilon_i^{jk}}{\sqrt{g}} \mathcal{F}_{jk}, \quad \nabla^i B_i = 0, \\ \mathcal{G} = a_0 + a_1 \zeta_1 + a_2 \zeta_1^2 + a_3 \zeta_1^3 + a_4 \zeta_2 + a_5 \zeta_1 \zeta_2 \\ + a_6 \zeta_3 + a_7 \zeta_4, \quad (3.4)$$

with a_n being arbitrary functions of $A^k A_k$ only, and

$$\zeta_1 = B_i B^i, \quad \zeta_2 = (\nabla_i B_j) (\nabla^i B^j), \\ \zeta_3 = (\nabla_i B_j) (\nabla^i B^k) (\nabla^j B_k), \\ \zeta_4 = (\nabla_i \nabla_j B_k) (\nabla^i \nabla^j B^k). \quad (3.5)$$

In terms of the magnetic field B_i , the 3-tensor \mathcal{F}_{ij} can be written as

$$\mathcal{F}_{ij} \equiv \frac{\varepsilon_{ij}^k}{\sqrt{g}} B_k. \quad (3.6)$$

Variations of S_{EM} with respect to A_0 and A_i yield the generalized Maxwell equations, given respectively, by

$$\nabla^k \mathcal{F}_{0k} = \frac{1}{2} m_e^2 (A_0 - \hat{N}^i A_i), \quad (3.7) \\ \frac{1}{N\sqrt{g}} \partial_t \left[\frac{\sqrt{g}}{N} g^{ij} (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) \right] = -\frac{1}{N^2} \left\{ g^{kj} \nabla_k [\hat{N}^i \right. \\ \times (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl})] - g^{ij} \nabla_k [\hat{N}^k (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl})] \Big\} \\ + \frac{m_e^2}{2N^2} (A_0 - \hat{N}^j A_j) \hat{N}^i + \frac{1}{2} A^i \left(a'_0 + a'_1 \zeta_1 + a'_2 \zeta_1^2 \right. \\ \left. + a'_3 \zeta_1^3 + a'_4 \zeta_2 + a'_5 \zeta_1 \zeta_2 + a'_6 \zeta_3 + a'_7 \zeta_4 \right) \\ + \frac{\varepsilon_k^{ji}}{2\sqrt{g}} \nabla_j \left[(a_1 + 2a_2 \zeta_1 + 3a_3 \zeta_1^2 + a_5 \zeta_2) B^k \right] \\ + \frac{\varepsilon_j^{ik}}{4\sqrt{g}} \nabla_k \nabla_l \left\{ 2(a_4 + a_5 \zeta_1) \nabla^l B^j \right. \\ \left. + a_6 \left[(\nabla^l B^m) (\nabla^j B_m) + (\nabla^l B_m) (\nabla^m B^j) \right. \right. \\ \left. \left. + (\nabla_m B^l) (\nabla^m B^j) \right] \right\}$$

$$+ \frac{\varepsilon_k^{li}}{2\sqrt{g}} \nabla_l \nabla_j \nabla_m \left(a_7 \nabla^m \nabla^j B^k \right) \\ - \frac{2g_e^2}{N} \mathcal{K}' A^i B_j B^j (A - \mathcal{A}) \\ + \frac{g_e^2 \varepsilon_j^{ik}}{N\sqrt{g}} \nabla_k \left[\mathcal{K} B^j (A - \mathcal{A}) \right] \\ - \frac{g_e^2 \varepsilon_k^{ji}}{N\sqrt{g}} \nabla_j \left[\mathcal{K} B^k (A - \mathcal{A}) \right], \quad (3.8)$$

where a prime denotes the ordinary derivative with respect to the indicated argument.

On the other hand, when the electromagnetic field is the only source, we find that J^t, J_i, J_φ, J_A and τ_{ij} are given by Eq.(B.1) in Appendix B.

IV. SPHERICAL STATIC SPACETIMES FILLED WITH AN ELECTROMAGNETIC FIELD

Spherically symmetric static vacuum spacetimes with projectability condition in the HMT setup were studied systematically in [21, 30–32]. In particular, the metric can be cast in the form,

$$ds^2 = -c^2 dt^2 + e^{2\nu} (dr + e^{\mu-\nu} dt)^2 + r^2 d^2\Omega, \quad (4.1)$$

in the spherical coordinates $x^i = (r, \theta, \phi)$, where $d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2$, and

$$\mu = \mu(r), \quad \nu = \nu(r), \quad N^i = e^{\mu-\nu} \delta_r^i. \quad (4.2)$$

The corresponding timelike Killing vector is $\xi = \partial_t$. With the gauge freedom of the Newtonian prepotential and the electromagnetic field, without loss of the generality, we choose the gauges,

$$\varphi = 0, \quad A_i(r) = A_1(r) \delta_i^r, \quad (4.3)$$

that is, we consider only the electric field and set the magnetic field to zero. Then, we find that

$$\mathcal{L}_\varphi = 0, \quad F_\varphi^{ij} = 0, \quad B_i = 0, \\ \pi_{ij} = -e^{\mu+\nu} \left(\mu' \delta_i^r \delta_j^r + r e^{-2\nu} \Omega_{ij} \right) \\ + e^{\mu-\nu} g_{ij} \left(\mu' + \frac{2}{r} \right), \\ K_{ij} = e^{\mu+\nu} \left(\mu' \delta_i^r \delta_j^r + r e^{-2\nu} \Omega_{ij} \right), \\ R_{ij} = \frac{2\nu'}{r} \delta_i^r \delta_j^r + e^{-2\nu} \left[r\nu' - (1 - e^{2\nu}) \right] \Omega_{ij}, \\ \mathcal{L}_K = -\frac{2}{r^2} e^{2(\mu-\nu)} (2r\mu' + 1), \\ \mathcal{L}_A = \frac{2A}{r^2} \left[e^{-2\nu} (1 - 2r\nu') + (\Lambda_g r^2 - 1) \right], \quad (4.4)$$

where $\Omega_{ij} \equiv \delta_i^\theta \delta_j^\theta + \sin^2\theta \delta_i^\phi \delta_j^\phi$, and \mathcal{L}_V is too complicated to be given explicitly here. Then, the Hamiltonian constraint (2.8) reads,

$$\int (\mathcal{L}_K + \mathcal{L}_V - 8\pi G J^t) e^\nu r^2 dr = 0, \quad (4.5)$$

where

$$J^t = -\frac{1}{2g_e^2} \left[2A_0'^2 e^{-2\nu} + m_e^2 (A_0 - A_1 e^{\mu-\nu})^2 \right]. \quad (4.6)$$

The momentum constraint (2.10) reduces to,

$$\nu' e^\mu = -\frac{2\pi G m_e^2}{g_e^2} (A_0 e^\nu - A_1 e^\mu) r A_1, \quad (4.7)$$

where

$$J^i = \frac{m_e^2}{2g_e^2} (A_0 - A_1 e^{\mu-\nu}) A_1 e^{-2\nu} \delta_r^i. \quad (4.8)$$

Eqs.(2.12) and (2.13), on the other hand, now read, respectively,

$$\begin{aligned} & \left[e^{2\nu} (\Lambda_g r^2 - 1) + 1 \right] e^{\mu+\nu} \mu' - 2 \left(\nu' - \Lambda_g r e^{2\nu} \right) e^{\mu+\nu} \\ &= 8\pi G r^2 e^{4\nu} J_\varphi, \end{aligned} \quad (4.9)$$

$$2r\nu' - \left[e^{2\nu} (\Lambda_g r^2 - 1) + 1 \right] = 0, \quad (4.10)$$

where $J_A = 0$, and

$$\begin{aligned} J_\varphi = & \frac{m_e^2 e^{-2\nu}}{2g_e^2} \left[(A_1 A_0)' - 2A_1 A_1' e^{\mu-\nu} \right. \\ & - A_1^2 e^{\mu-\nu} \left(\mu' - 2\nu' + \frac{2}{r} \right) \\ & \left. - A_0 A_1 \left(\nu' - \frac{2}{r} \right) \right]. \end{aligned} \quad (4.11)$$

The dynamical equations (2.15) yield,

$$\begin{aligned} & e^{2\mu} \left[2 \left(\mu' + \nu' \right) + \frac{1}{r} \right] + \frac{1}{2} r e^{2\nu} \mathcal{L}_A = -r \left(F_{rr} \right. \\ & \left. + F_{rr}^A + 8\pi G \tau_{rr} \right), \end{aligned} \quad (4.12)$$

$$\begin{aligned} & e^{2\mu} \left[\mu'' + (2\mu' - \nu') \left(\mu' + \frac{1}{r} \right) \right] + \frac{1}{2} e^{2\nu} \mathcal{L}_A \\ &= -\frac{e^{2\nu}}{r^2} \left(F_{\theta\theta} + F_{\theta\theta}^A + 8\pi G \tau_{\theta\theta} \right), \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \tau_{rr} = & -\frac{1}{4g_e^2} \left[\mathcal{G} e^{2\nu} + 2A_0'^2 - 2a_0' A_1^2 \right. \\ & \left. - m_e^2 e^{2\nu} (A_0 - A_1 e^{\mu-\nu})^2 \right], \\ \tau_{\theta\theta} = & -\frac{r^2 e^{-2\nu}}{4g_e^2} \left[\mathcal{G} e^{2\nu} - 2A_0'^2 \right. \\ & \left. - m_e^2 e^{2\nu} (A_0 - A_1 e^{\mu-\nu})^2 \right]. \end{aligned} \quad (4.14)$$

The Maxwell equations (3.7) and (3.8) now become,

$$a_0' A_1 + m_e^2 e^{\mu+\nu} (A_0 - A_1 e^{\mu-\nu}) = 0, \quad (4.15)$$

$$A_0'' - A_0' \nu' + \frac{2}{r} A_0' - \frac{1}{2} m_e^2 e^{2\nu} (A_0 - e^{\mu-\nu} A_1) = 0. \quad (4.16)$$

A. Massless Electromagnetic Field with $N^r \neq 0$

When $N^r \neq 0$ (or $\mu \neq -\infty$), for a massless electromagnetic field, Eq.(4.7) immediately gives $\nu = \nu_0$, where ν_0 is an integration constant. Then, Eq.(4.10) yields,

$$\nu = 0 = \Lambda_g. \quad (4.17)$$

When $m_e = 0$, Eqs.(4.15) and (4.16) yield,

$$A_0 = \frac{Q}{r} + Q_0, \quad A_1 = 0, \quad (4.18)$$

where Q and Q_0 are two integration constant. Without loss of generality, we can always set $Q_0 = 0$. It is remarkable that the above solution for the electromagnetic field is the same as that given in GR. On the other hand, when $\nu = 0$, the spatial part is flat, $R_{ij} = 0$, and all the high order spatial derivative terms vanish, so we have $F^{ij} = -\Lambda g_{ij}$. Then, Eqs.(4.12) and (4.13) reduce to,

$$\left(2\mu' + \frac{1}{r} \right) e^{2\mu} = \Lambda r^2 - 2rA' + \frac{4\pi G Q^2}{g_e^2 r^2}, \quad (4.19)$$

$$\begin{aligned} & \left[\mu'' + 2\mu' \left(\mu' + \frac{1}{r} \right) \right] e^{2\mu} = \frac{1}{r} \left[\Lambda r - (rA')' \right. \\ & \left. - \frac{4\pi G Q^2}{g_e^2 r^3} \right]. \end{aligned} \quad (4.20)$$

Note that these two equations are not independent. In fact, one can obtain Eq.(4.20) from Eq.(4.19). Therefore, we have one equation for two unknowns, μ and A . Thus, similar to the vacuum case [31], for any chosen gauge field A , the metric coefficient μ is given by

$$\begin{aligned} \mu = & \frac{1}{2} \ln \left(\frac{2m}{r} + \frac{1}{3} \Lambda r^2 - \frac{4\pi G Q^2}{g_e^2 r^2} \right. \\ & \left. - 2A + \frac{2}{r} \int^r A(r') dr' \right). \end{aligned} \quad (4.21)$$

Then, the Hamiltonian constraint (4.5) reduces to

$$\int_0^\infty A' r dr = 0. \quad (4.22)$$

Therefore, for any given A , the solutions of Eqs.(4.17), (4.18) and (4.21) represent solutions of the HL theory in the HMT setup, coupled with an electromagnetic field, provided that Eq.(4.22) is satisfied.

When $A = \text{Constant}$, the above solutions reduce exactly to the Reissner-Nordström solution found in GR but written in the Painleve-Gullstrand coordinates [33].

B. Massless Electromagnetic Field with $N^r = 0$

In the diagonal case, we have

$$N^r = 0, \quad \text{or} \quad \mu = -\infty. \quad (4.23)$$

Then, $K_{ij} = 0 = \pi_{ij}$, for which the momentum constraint is satisfied identically, while Eq.(4.10) becomes,

$$\nu' = \frac{1}{2r} \left[e^{2\nu} (\Lambda_g r^2 - 1) + 1 \right], \quad (4.24)$$

which has the general solution,

$$\nu = -\frac{1}{2} \ln \left(1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2 \right), \quad (4.25)$$

where M is a constant. Then, the Maxwell equations (4.15) and (4.16) reduce to,

$$A_1 = 0, \quad (4.26)$$

$$A_0'' + A_0' \left(\frac{2}{r} - \nu' \right) = 0. \quad (4.27)$$

Eq.(4.27) has the general solution,

$$A_0 = D_1 \int \frac{dr}{r^4 \left(1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2 \right)} + D_2, \quad (4.28)$$

where D_1 and D_2 are two integration constants. For the solutions μ and ν , given by Eqs.(4.23) and (4.25), it can be shown that only one of the two dynamical equations (4.12) and (4.13) is independent, and can be cast in the form,

$$A' + P(r)A = Q(r), \quad (4.29)$$

where

$$\begin{aligned} P(r) &= \frac{\Lambda_g r^3 - 3M}{r [3(r - 2M) - \Lambda_g r^3]}, \\ Q(r) &= \frac{1}{2 [3(r - 2M) - \Lambda_g r^3]} \left(\frac{4\pi G D_1^2}{g_e^2 r^2} \right. \\ &\quad \left. - \alpha_1 r^2 - \frac{\alpha_2}{r} - \frac{\alpha_3}{r^4} - \frac{\alpha_4}{r^6} - \frac{\alpha_5}{r^7} \right), \end{aligned} \quad (4.30)$$

with

$$\begin{aligned} \alpha_1 &= -\frac{3}{2} g_0 \zeta^2 + \Lambda_g + \left(2g_2 + \frac{2}{3} g_3 \right) \Lambda_g^2 \zeta^{-2} \\ &\quad + \left(12g_4 + 4g_5 + \frac{4}{3} g_6 \right) \Lambda_g^3 \zeta^{-4}, \\ \alpha_2 &= 6M - (24g_2 + 10g_3) M \Lambda_g \zeta^{-2} \\ &\quad - (72g_4 + 28g_5 + 12g_6 - 2g_8) M \Lambda_g^2 \zeta^{-4}, \\ \alpha_3 &= -3g_3 M^2 \zeta^{-2} + (78g_5 + 90g_6 - 75g_8) M^2 \Lambda_g \zeta^{-4}, \\ \alpha_4 &= 27M^2 \zeta^{-4} (7g_8 - 9g_6 - 8g_5), \\ \alpha_5 &= -18M^3 \zeta^{-4} (20g_8 - 25g_6 - 22g_5). \end{aligned} \quad (4.31)$$

The general solution of Eq.(4.29) is given by,

$$\begin{aligned} A(r) &= e^{-\int^r P(r') dr'} \\ &\quad \times \left(\int^r Q(r') e^{\int^{r'} P(r'') dr''} dr' + C_A \right) \end{aligned}$$

$$\begin{aligned} &= \sqrt{1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2} \\ &\quad \times \left(C_A - \frac{1}{6} \int^r \mathcal{D}(r') dr' \right), \end{aligned} \quad (4.32)$$

where C_A is an integration constant, and

$$\begin{aligned} \mathcal{D}(r) &\equiv \frac{1}{\sqrt{r^{16} \left(1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2 \right)^3}} \left(\alpha_1 r^9 + \alpha_2 r^6 \right. \\ &\quad \left. + \alpha_3 r^3 + \alpha_4 r + \alpha_5 - \frac{4\pi G D_1^2}{g_e^2} r^5 \right). \end{aligned} \quad (4.33)$$

For the special case $\Lambda_g = 0$, A is given explicitly by

$$\begin{aligned} A(r) &= 1 + \frac{D_1^2}{12g_e^2 M^2} \left(\frac{M}{r} - 1 \right) + C_A \sqrt{1 - \frac{2M}{r}} \\ &\quad + \frac{g_0 \zeta^2}{8} \left[r^2 + 5Mr - 30M^2 \left(1 \right. \right. \\ &\quad \left. \left. - \sqrt{1 - \frac{2M}{r}} \ln \left(\sqrt{r} + \sqrt{r - 2M} \right) \right) \right] \\ &\quad + \frac{g_3}{10M^2 r^3 \zeta^2} (M^3 + M^2 r + 2Mr^2 - 2r^3) \\ &\quad - \frac{\alpha_4}{378M^6 r^5} \left(7M^5 + 5M^4 r + 4M^3 r^2 \right. \\ &\quad \left. + 4M^2 r^3 + 8Mr^4 - 8r^5 \right) \\ &\quad - \frac{\alpha_5}{1386M^7 r^6} \left(21M^6 + 14M^5 r + 10M^4 r^2 \right. \\ &\quad \left. + 8M^3 r^3 + 8M^2 r^4 + 16Mr^5 - 16r^6 \right). \end{aligned} \quad (4.34)$$

V. CHARGED BLACK HOLES

The causal structure of spacetimes in the HL theory is different from that in GR, because of the breaking of the Lorentz symmetry. In particular, the dispersion relations of particles contain high-order momentum terms [1, 9, 34, 35],

$$\omega_k^2 = m^2 + k^2 \left(1 + \frac{k^2}{M_A^2} + \frac{k^4}{M_B^4} \right), \quad (5.1)$$

where M_A and M_B are the suppression energy scales of the fourth and sixth order derivative terms. As a result, the speeds of particles v_p ($\equiv d\omega_k/dk$) become unbounded in the UV, and their motions do not follow geodesics. This immediately makes all the definitions of black holes given in GR invalid [22–25]. To provide a proper definition of black holes, anisotropic conformal boundaries

[36] and kinematics of particles [37] have been studied in the HL theory. In particular, in [21] black holes and global structure of spacetimes were studied by defining a horizon as the infinitely redshifted 2-dimensional (closed) surface of massless test particles [29]. Such a definition reduces to that given in GR when the dispersion relation is relativistic, where $M_A, M_B \gg k$, as one can see from Eq.(5.1).

To study the black hole structure of the solutions presented in the last section, following [21] let us consider a scalar field with a given dispersion relation $F(\zeta)$. In the geometrical optical approximations, ζ is given by $\zeta = g_{ij}k^i k^j$, where k_i denotes the 3-momentum of the corresponding massless particle. With this approximation, the trajectory of a test particle is given by

$$S_p \equiv \int_0^1 \mathcal{L}_p d\tau = \frac{1}{2} \int_0^1 d\tau \left\{ \frac{c^2 N^2}{e} \dot{t}^2 + e [F(\zeta) - 2\zeta F'(\zeta)] \right\}, \quad (5.2)$$

where e is a one-dimensional einbein, and ζ is now considered as a functional of $t, x^i, \dot{t}, \dot{x}^i$ and e , given by the relation,

$$\zeta [F'(\zeta)]^2 = \frac{1}{e^2} g_{ij} (\dot{x}^i + N^i \dot{t}) (\dot{x}^j + N^j \dot{t}), \quad (5.3)$$

with $\dot{t} \equiv dt/d\tau$, etc. Considering Eq.(5.1), we assume that

$$F(\zeta) = \zeta^n, \quad (n = 1, 2, \dots). \quad (5.4)$$

Then, Eq.(5.3) yields,

$$\zeta = \left(\frac{\dot{r} + N^r \dot{t}}{ne\sqrt{f}} \right)^{2/(2n-1)} \equiv \left(\frac{\mathcal{D}}{e^2} \right)^{1/(2n-1)}, \quad (5.5)$$

where

$$N^r = -e^{\mu-\nu}, \quad f = e^{-2\nu}. \quad (5.6)$$

Note that there is a sign difference between N^r defined here and the one defined in Eq.(4.2). This corresponds to the coordinate transformation $t \rightarrow -t$, that is, in the static case, if (N, N^r, ν) is a solution of the HL theory, so is the one $(N, -N^r, \nu)$. Keeping this in mind, and inserting the above into Eq.(5.2), we find that, for radially moving massless particles, \mathcal{L}_p is given by

$$\mathcal{L}_p = \frac{N^2}{2e} \dot{t}^2 + \frac{1}{2} (1 - 2n) e^{1/(1-2n)} \mathcal{D}^{n/(2n-1)}. \quad (5.7)$$

Then, the variations of $\mathcal{L}_p = 0$ with respect to e and t yield, respectively,

$$N^2 \dot{t}^2 - e^{2(n-1)/(2n-1)} \mathcal{D}^{n/(2n-1)} = 0, \quad (5.8)$$

$$N^2 \dot{t} - e^{2(n-1)/(2n-1)} \frac{N^r}{\sqrt{f}} \mathcal{D}^{1/[2(2n-1)]} = eE, \quad (5.9)$$

where E is an integration constant, representing the total energy of the test particle. eliminating e from Eqs.(5.8) and (5.9) we find that

$$X^n - p(r)X - q(r, E) = 0, \quad (5.10)$$

where

$$X \equiv \left(\frac{\sqrt{\mathcal{D}}}{\dot{t}} \right)^{1/(n-1)} = \left(\frac{|r' + N^r|}{n\sqrt{f}} \right)^{1/(n-1)},$$

$$p(r) \equiv \frac{N^r}{\sqrt{f}}, \quad q(r, E) \equiv EN^{1/(n-1)}, \quad (5.11)$$

with $r' \equiv \dot{r}/\dot{t} = dr/dt$. Once X is found by solving Eq.(5.11), from it we obtain

$$t = t_0 + \int \frac{dr}{H(r, E)}, \quad (5.12)$$

where

$$H(r, E) = \epsilon n e^{-\nu} X^{n-1} - N^r$$

$$= (\epsilon n - 1) N^r + \epsilon n E \frac{N^{1/(n-1)}}{X} e^{-\nu}, \quad (5.13)$$

with $\epsilon = \text{sign}(\dot{r} + N^r \dot{t})$. In the last step of the above expressions, we used Eq.(5.10) to replace X^{n-1} . For detail, we refer readers to [21].

A horizon is defined as a surface that is free of space-time singularities and on which massless test particles are infinitely redshifted. Note that the nature of singularities in the HL theory was studied in [38], and was shown that they can be classified into two classes: the coordinate singularities and spacetime singularities. The coordinate singularities are the ones that can be removed by the general coordinate transformations (1.1), while the spacetime singularities are ones that cannot be removed by (1.1). It should be noted that, although these definitions are the same as those given in GR, there are fundamental differences, because of the symmetry (1.1) of the HL theory. In [38], some examples are given in which coordinate singularities in GR become spacetime singularities in the HL theory. Spacetime singularities can be further divided into two kinds: the curvature and non-curvature ones. A curvature singularity is defined as the one in which at least one of the scalars of the symmetry (1.2) is singular. A non-curvature singularity is defined as the one that does not have curvature singularity, but some other physical quantities, such as tidal forces and/or distortions experienced by a test particle, become unbounded.

Assuming that at a surface, say, r_H , H defined by Eq.(5.13) behaves as

$$H(r, E) = H_0(r_H, E) (r - r_H)^\delta + \dots, \quad (5.14)$$

as $r \rightarrow r_H$, where $H_0(r_H, E) \neq 0$. Then,

$$H'(r, E) \Big|_{r=r_H} = \begin{cases} 0, & \delta > 1, \\ H_0(r_H, E), & \delta = 1, \\ \pm\infty, & 0 < \delta < 1. \end{cases} \quad (5.15)$$

Now $t \rightarrow \infty$ as $r \rightarrow r_H^\pm$ if and only if

$$\delta \geq 1, \quad (5.16)$$

for which we have

$$\left. \frac{dH(r, E)}{dr} \right|_{r=r_H} = \text{finite}. \quad (5.17)$$

This provides the necessary condition on f, N, N^r, E, n for the hypersurface r_H to be a horizon, as defined above. It should be noted that r_H usually depends on the energy E of the test particles, as first noted in [21].

To study the black hole structure of the solutions presented in the last section, let us consider the cases $N^r = 0$ and $N^r \neq 0$ separately.

A. $N^r \neq 0$

In this case, the solutions are given by Eqs.(4.17) and (4.21). Inserting them into Eq.(5.14) and considering the Eq.(5.6), we find that

$$H(r, E) = (1+n)e^\mu - \frac{nE}{X}. \quad (5.18)$$

Note that in writing the above expression, we had chosen $\epsilon = -1$ [21]. Thus, at $r = r_H$, we have

$$X(r_H, E) = \frac{nE}{1+n} e^{-\mu(r_H)}. \quad (5.19)$$

Inserting it into Eq.(5.10), we obtain

$$\mu(r) - \mu_{(n,E)} = 0, \quad (5.20)$$

at $r = r_H$, where

$$\mu_{(n,E)} \equiv \frac{1}{n} \ln \left[\frac{1}{n} \left(\frac{1+n}{nE} \right)^{n-1} \right]. \quad (5.21)$$

On the other hand, from Eqs.(5.10) and (5.18), we also have

$$H'(r_H, E) = \frac{1}{2}(1+n)e^{\mu_{(n,E)}} \mu'(r_H). \quad (5.22)$$

Thus, the condition (5.17) requires

$$\mu'(r)|_{r=r_H} = \text{finite}. \quad (5.23)$$

Inserting the general solution (4.21) into Eq.(5.20), one can find all the roots, $r = r_H$. Provided that the solutions have no spacetime singularities and the condition (5.23) holds, the 2-sphere $r = r_H$ represents a horizon. As mentioned above, r_H in general depends on E , that is, the radius of the horizon is observer-dependent. Such a dependence is the reflection of the fact that the HL theory breaks the Lorentz symmetry.

To see the above explicitly, let us consider the case $A = \text{Constant}$ and $\Lambda = 0$, for which Eq.(4.21) reduces to Reissner-Nordström solution found in GR but written in the Painleve-Gullstrand coordinates [33],

$$\mu_{RN}(r) = \frac{1}{2} \ln \left(\frac{r_g}{r} - \frac{r_Q^2}{r^2} \right), \quad (5.24)$$

where $r_g \equiv 2m$, $r_Q^2 \equiv 4\pi G Q^2 / g_e^2$. Inserting it into Eq.(5.20) we find that

$$r_H^\pm = \frac{1}{2} e^{-2\mu_{(n,E)}} \left(r_g \pm \sqrt{r_g^2 - 4e^{2\mu_{(n,E)}} r_Q^2} \right). \quad (5.25)$$

It can be shown that the solutions at r_H^\pm is free of space-time singularities and Eq.(5.23) is satisfied. Therefore, provided that $r_g^2 - 4r_Q^2 e^{2\mu_{(n,E)}} \geq 0$, the solutions have two horizons at $r = r_H^\pm$. When $n = 1$, we have $\mu_{(n=1,E)} = 0$, and the above expressions reduce exactly to those given in GR [39]. However, when $n > 1$, from Eq.(5.21) we find that

$$\mu_{(n,E)} \simeq -\frac{1+n}{n} \ln E, \quad (5.26)$$

as $E \rightarrow \infty$. Then, Eq.(5.25) show that $r_H^\pm \simeq 0$, that is, as long as the test particles have enough energy ($E \gg 1$), the horizons can be as closed to the singularity at $r = 0$ as desired. A similar situation also happens to the Schwarzschild solution [21]. Again, this is because of the violation of the Lorentz symmetry in the UV regime.

When $A \neq \text{Constant}$, the local and global structures of the corresponding spacetimes depend on the choice of $A(r)$ (as well as Λ). It is not difficult to see that the spacetimes have very rich structures, and some of them will quite similar to the Reissner-Nordström solution, and in general the radius r_H will depend on the energy of the observers, i.e., $r_H = r_H(E)$.

B. $N^r = 0$

When $N^r = 0$, we have $X = E^{1/n}$ and

$$\begin{aligned} H(r, E) &= \epsilon n E^{(n-1)/n} e^{-\nu(r)} \\ &= \epsilon n E^{(n-1)/n} \sqrt{1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2}. \end{aligned} \quad (5.27)$$

To have a horizon, the necessary condition (5.16) requires that

$$1 - \frac{2M}{r} - \frac{\Lambda_g}{3} r^2 = 0, \quad (5.28)$$

has at least two equal and positive roots. It is easy to show that this is not possible for any choice of Λ_g and M . Therefore, this class of solutions does not have black hole structures. The global structures of the spacetimes for various choices of the free parameters Λ_g and M are given in [21], and we shall not repeat these analyses here.

VI. CONCLUSIONS

In this paper, we have studied electromagnetic static spacetimes in the Hořava and Melby-Thompson (HMT) setup [11], in which $\lambda = 1$. After writing down specifically the coupling of the theory with a massive vector field in Sec. III, we have applied the general formulas to electric static spacetimes with spherical symmetry in Sec. IV, and found all the solutions of the massless vector field under the gauge (4.3). In particular, when $N^r \neq 0$, the metric coefficients are given by Eqs.(4.17) and (4.21) for any given gauge field A , subjected to the constraint (4.22). The corresponding electromagnetic field $(A_0, A_1, 0, 0)$ is given by Eq.(4.18). When $A = \text{Constant}$, the solutions reduce to the Reissner-Nordström one, found in GR but written in the Painleve-Gullstrand coordinates [33].

In the diagonal case $N^r = 0$, the metric coefficients are given by Eqs.(4.23) and (4.25), while the corresponding electromagnetic field $(A_0, A_1, 0, 0)$ and the gauge field A are given, respectively, by Eqs.(4.26), (4.28) and (4.32). When $\Lambda_g = 0$, the gauge field A is explicitly given by Eq.(4.34).

In Sec. V, using the geometrical optical approximations [21], we have studied the existence of horizons and shown explicitly why the definitions of black holes given in GR [22–25] cannot be applied to the HL gravity, and how to generalize those definitions to the HL theory, because of the breaking of the Lorentz-invariance, in which the conception of light-cones is no longer valid. Applying the new definition of horizons to our solutions presented in Sec. IV, we have found that some of the solutions with $N^r \neq 0$ give rise to black holes, although the locations of the horizons depend on the energies of the test particles. With sufficient high energy, the horizon can be as closed to the central singularity at origin as desired. On the other hand, the solutions with $N^r = 0$ do not have black hole structures. It should be noted that one might argue that by the coordinate transformations,

$$d\tilde{r} = \frac{dr}{N^r(r)} + dt, \quad (6.1)$$

one can bring the metric with $N^r \neq 0$ into the diagonal form,

$$ds^2 = -dt^2 + e^{2\tilde{\nu}} d^2\tilde{r} + r^2 d^2\Omega, \quad (6.2)$$

where $\tilde{\nu} = \nu(r) + \ln|N^r(r)|$. However, now r is time-dependent, $r = r(\tilde{r}, t)$ (and so is $\tilde{\nu}$), and then the trajectories of massless particles are no longer described by Eqs.(5.8) and (5.9), so the discussions presented in Sec. V cannot be applied to the “dynamical” case.

In addition, the gauge field A for the non-diagonal case (See Sec. IV.A) is undetermined. While its physics is not clear (even in more general case) [11, 12], the solar system tests generically require it vanish [31] (See also [32]). However, in the diagonal case it is uniquely determined by Eq.(4.32). HMT showed that this class of

solutions is consistent with observations only when the gauge field A is considered as part of the lapse function in the IR, $\mathcal{N} = N - A$. In [31] a different point of view was adopted, in which the gauge field as well as the Newtonian prepotential was considered as independent of the spacetime metric, although they are part of the gravitational field and interact with spacetime through the field equations, roles quite similar to the Brans-Dicke scalar field in the Brans-Dicke theory of gravity [40]. This is seemingly supported by the results presented recently in [41]. Moreover, some preliminary results of stability analysis show that this class of solutions might not be stable [42]. Clearly, to understand these solutions better, further investigations are highly demanded, including the possibilities of considering them as describing the spacetime outside of a charged star. The general junction conditions across the surface of a star have been worked out in [31], although internal solutions of charged stars have not been found, yet.

Acknowledgements: The work of AW was supported in part by DOE Grant, DE-FG02-10ER41692. KL was partially supported by NSFC No. 11178018 and No. 11075224.

Appendix A: $(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$

$(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$ appearing in Eq.(2.16) are given, respectively,

$$\begin{aligned} (F_0)_{ij} &= -\frac{1}{2}g_{ij}, \\ (F_1)_{ij} &= -\frac{1}{2}g_{ij}R + R_{ij}, \\ (F_2)_{ij} &= -\frac{1}{2}g_{ij}R^2 + 2RR_{ij} - 2\nabla_{(i}\nabla_{j)}R \\ &\quad + 2g_{ij}\nabla^2 R, \\ (F_3)_{ij} &= -\frac{1}{2}g_{ij}R_{mn}R^{mn} + 2R_{ik}R_j^k - 2\nabla^k\nabla_{(i}R_{j)k} \\ &\quad + \nabla^2 R_{ij} + g_{ij}\nabla_m\nabla_n R^{mn}, \\ (F_4)_{ij} &= -\frac{1}{2}g_{ij}R^3 + 3R^2R_{ij} - 3\nabla_{(i}\nabla_{j)}R^2 \\ &\quad + 3g_{ij}\nabla^2 R^2, \\ (F_5)_{ij} &= -\frac{1}{2}g_{ij}RR^{mn}R_{mn} + R_{ij}R^{mn}R_{mn} \\ &\quad + 2RR_{ki}R_j^k - \nabla_{(i}\nabla_{j)}(R^{mn}R_{mn}) \\ &\quad - 2\nabla^n\nabla_{(i}RR_{j)n} + g_{ij}\nabla^2(R^{mn}R_{mn}) \\ &\quad + \nabla^2(RR_{ij}) + g_{ij}\nabla_m\nabla_n(RR^{mn}), \\ (F_6)_{ij} &= -\frac{1}{2}g_{ij}R_n^m R_p^p R_m^p + 3R^{mn}R_{ni}R_{mj} \\ &\quad + \frac{3}{2}\nabla^2(R_{in}R_j^n) + \frac{3}{2}g_{ij}\nabla_k\nabla_l(R_n^k R^{ln}) \\ &\quad - 3\nabla_k\nabla_{(i}(R_{j)n}R^{nk}), \end{aligned}$$

$$\begin{aligned}
(F_7)_{ij} &= -\frac{1}{2}g_{ij}(\nabla R)^2 + (\nabla_i R)(\nabla_j R) - 2R_{ij}\nabla^2 R \\
&\quad + 2\nabla_{(i}\nabla_{j)}\nabla^2 R - 2g_{ij}\nabla^4 R, \\
(F_8)_{ij} &= -\frac{1}{2}g_{ij}(\nabla_p R_{mn})(\nabla^p R^{mn}) - \nabla^4 R_{ij} \\
&\quad + (\nabla_i R_{mn})(\nabla_j R^{mn}) + 2(\nabla_p R_{in})(\nabla^p R_{jn}^n) \\
&\quad + 2\nabla^n \nabla_{(i}\nabla^2 R_{j)n} + 2\nabla_n (R_{ij}^n \nabla_{(i} R_{j)}^m) \\
&\quad - 2\nabla_n (R_{m(j}\nabla_{i)} R^{mn}) - 2\nabla_n (R_{m(i}\nabla^n R_{j)}^m) \\
&\quad - g_{ij}\nabla^n \nabla^m \nabla^2 R_{mn}, \tag{A.1}
\end{aligned}$$

and [12]

$$\begin{aligned}
F_{(\varphi,1)}^{ij} &= \frac{1}{2}\varphi \left\{ (2K + \nabla^2 \varphi) R^{ij} - 2(2K_k^j + \nabla^j \nabla_k \varphi) R^{ik} \right. \\
&\quad \left. - 2(2K_k^i + \nabla^i \nabla_k \varphi) R^{jk} - (2\Lambda_g - R)(2K^{ij} + \nabla^i \nabla^j \varphi) \right\}, \\
F_{(\varphi,2)}^{ij} &= \frac{1}{2}\nabla_k \left\{ \varphi \mathcal{G}^{ik} \left(\frac{2N^j}{N} + \nabla^j \varphi \right) \right. \\
&\quad \left. + \varphi \mathcal{G}^{jk} \left(\frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi \mathcal{G}^{ij} \left(\frac{2N^k}{N} + \nabla^k \varphi \right) \right\}, \\
F_{(\varphi,3)}^{ij} &= \frac{1}{2} \left\{ 2\nabla_k \nabla^{(i} f_{\varphi}^{j)k} - \nabla^2 f_{\varphi}^{ij} - (\nabla_k \nabla_l f_{\varphi}^{kl}) g^{ij} \right\}, \tag{A.2}
\end{aligned}$$

where

$$f_{\varphi}^{ij} = \varphi \left\{ (2K^{ij} + \nabla^i \nabla^j \varphi) - \frac{1}{2}(2K + \nabla^2 \varphi) g^{ij} \right\}. \tag{A.3}$$

Appendix B: $J^t, J_i, J_{\varphi}, J_A$ and τ_{ij} for a vector field

When a vector field is the only source, the quantities $J^t, J_i, J_{\varphi}, J_A$ and τ_{ij} are given by

$$\begin{aligned}
J^t &= \frac{1}{g_e^2} \left[-\frac{1}{N^2} g^{ij} (\mathcal{F}_{0i} - \hat{N}^k \mathcal{F}_{ki}) (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) \right. \\
&\quad + \frac{1}{N} g^{ij} (\mathcal{F}_{0i} - \hat{N}^k \mathcal{F}_{ki}) (\nabla^l \varphi \mathcal{F}_{jl}) \\
&\quad + \frac{1}{N} g^{ij} (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) (\nabla^k \varphi \mathcal{F}_{ki}) \\
&\quad + \frac{m_e^2}{N} (A_0 - \hat{N}^i A_i) (\nabla^j \varphi A_j) \\
&\quad \left. - \frac{m_e^2}{2N^2} (A_0 - \hat{N}^i A_i)^2 \right], \\
J_i &= -\frac{1}{2g_e^2 N} g^{jk} \left[(\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) \mathcal{F}_{ki} \right. \\
&\quad \left. + (\mathcal{F}_{0k} - \hat{N}^l \mathcal{F}_{lk}) \mathcal{F}_{ij} \right]
\end{aligned}$$

$$\begin{aligned}
&+ \frac{m_e^2}{2g_e^2 N} (A_0 - \hat{N}^k A_k) A_i, \\
J_{\varphi} &= -\frac{1}{\sqrt{g}N} \partial_t \left[\sqrt{g} \mathcal{K} B_i B^i \right] \\
&+ \frac{m_e^2}{2g_e^2 N} \nabla^l \left[(A_0 - \hat{N}^i A_i) A_l \right] \\
&- \frac{1}{N} \nabla_j \left[\mathcal{K} B_i B^i (N \nabla^j \varphi - N^j) \right] \\
&+ \frac{1}{2g_e^2 N} g^{ij} \nabla^l \left[(\mathcal{F}_{0i} - \hat{N}^k \mathcal{F}_{ki}) \mathcal{F}_{jl} \right. \\
&\quad \left. (\mathcal{F}_{0j} - \hat{N}^k \mathcal{F}_{jk}) \mathcal{F}_{li} \right], \\
J_A &= 2\mathcal{K} B_i B^i, \\
\tau^{mn} &= g^{mn} \mathcal{L}_{EM} \\
&- \frac{1}{2g_e^2 N^2} \left[(\mathcal{F}_0^m - \hat{N}_k \mathcal{F}^{km}) (\mathcal{F}_0^n - \hat{N}_l \mathcal{F}^{nl}) \right. \\
&\quad + (\mathcal{F}_0^n - \hat{N}_k \mathcal{F}^{kn}) (\mathcal{F}_0^m - \hat{N}_l \mathcal{F}^{ml}) \\
&\quad + N (\mathcal{F}_{0j} - \hat{N}^l \mathcal{F}_{jl}) (\mathcal{F}^{mj} \nabla^n \varphi + \mathcal{F}^{nj} \nabla^m \varphi) \\
&\quad \left. + N (\mathcal{F}_{0i} - \hat{N}^l \mathcal{F}_{li}) (\mathcal{F}^{im} \nabla^n \varphi + \mathcal{F}^{in} \nabla^m \varphi) \right] \\
&- \frac{m_e^2}{2g_e^2 N} (A_0 - \hat{N}^i A_i) (A^m \nabla^n \varphi + A^n \nabla^m \varphi) \\
&- \frac{1}{2g_e^2} \left\{ a_1 (B^m B^n - B_i B^i g^{mn}) \right. \\
&\quad - a'_0 A^m A^n - a'_1 A^m A^n B_j B^j \\
&\quad - a'_2 A^m A^n (B_j B^j)^2 \\
&\quad + 2a_2 B_l B^l (B^m B^n - B_i B^i g^{mn}) \\
&\quad - a'_3 A^m A^n (B_j B^j)^3 + 3a_3 (B_l B^l)^2 (B^m B^n \\
&\quad - B_i B^i g^{mn}) - a'_4 A^m A^n (\nabla_l B_j) \nabla^l B^j \\
&\quad - a_4 \left[(\nabla^m B_j) \nabla^n B^j + (\nabla_i B^m) \nabla^i B^n \right] \\
&\quad + \left[\nabla_i (a_4 \nabla^i B^j) \right] B_j g^{mn} - 2 \left[\nabla_i (a_4 \nabla^i B^n) \right] B^m \\
&\quad + \nabla_i \left[a_4 (B^{(m} \nabla^i B^{n)} + B^{(m} \nabla^n) B^i \right. \\
&\quad \left. - B^i \nabla^{(m} B^{n)}) \right] - a'_5 A^m A^n B_k B^k (\nabla_l B_j) \nabla^l B^j \\
&\quad + a_5 (\nabla_l B_k) (\nabla^l B^k) (B^m B^n - B_i B^i g^{mn}) \\
&\quad - a_5 B_k B^k \left[(\nabla^m B_j) \nabla^n B^j + (\nabla_j B^m) \nabla^j B^n \right] \\
&\quad + \left[\nabla_i (a_5 B_k B^k \nabla^i B^j) \right] B_j g^{mn} \\
&\quad - 2 \left[\nabla_i (a_5 B_k B^k \nabla^i B^n) \right] B^m \\
&\quad + \nabla_i \left[a_5 B_k B^k (B^{(m} \nabla^i B^{n)} + B^{(m} \nabla^n) B^i \right.
\end{aligned}$$

$$\begin{aligned}
& -B^i \nabla^m B^n) \Big] \\
& -a'_6 A^m A^n (\nabla_l B_j) (\nabla^l B^k) \nabla^j B_k \\
& +a_6 (\nabla_i B^j) (\nabla^i B^m) \nabla_j B^n \\
& -\frac{1}{2} a_6 \Big[(\nabla^n B^j) (\nabla^m B^k) \nabla_j B_k \\
& + (\nabla^m B^j) (\nabla^n B^k) \nabla_j B_k \Big] \\
& +\frac{1}{2} B^j g^{mn} \nabla_i (a_6 \alpha_j^i) \\
& -\frac{1}{2} \nabla_i \Big[a_6 \Big(B^i \alpha^{mn} + B^{(m} \alpha^{in)} - B^{(n} \alpha^{m)i} \Big) \Big] \\
& -a'_7 A^m A^n (\nabla_l \nabla_j B_k) (\nabla^l \nabla^j B^k) \\
& -a_7 \Big[(\nabla^m \nabla_j B_k) (\nabla^n \nabla^j B^k) \\
& + (\nabla_i \nabla^m B_k) (\nabla^i \nabla^n B^k) \\
& - (\nabla_i \nabla_j B^n) (\nabla^i \nabla^j B^m) \Big] \\
& -B^l g^{mn} \nabla_j \nabla_i (a_7 \nabla^i \nabla^j B_l) \\
& + \nabla_k \Big(\beta^{k(mn)} - \beta^{(mn)k} - \beta^{(mkn)} \Big) \Big\} \\
& -\frac{2}{N} \Big[\mathcal{K}' A^m A^n B_j B^j (A - \mathcal{A}) \\
& -\mathcal{K} \Big(B^m B^n - B_i B^i g^{mn} \Big) (A - \mathcal{A}) \\
& -\frac{N}{2} \mathcal{K} B_i B^i (\nabla^m \varphi) (\nabla^n \varphi) \Big], \tag{B.1}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_j^i & \equiv (\nabla^i B^k) (\nabla_j B_k) + (\nabla^i B^k) (\nabla_k B_j) \\
& + (\nabla_k B^i) (\nabla^k B_j), \\
\beta^{kil} & \equiv -\nabla_j \Big(a_7 \nabla^j \nabla^i B^k \Big) B^l - a_7 (\nabla^i \nabla^l B_j) \nabla^k B^j \\
& + a_7 (\nabla^i \nabla^j B^k) \nabla_j B^l. \tag{B.2}
\end{aligned}$$

-
- [1] P. Hořava, JHEP, **0903**, 020 (2009) [arXiv:0812.4287]; Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775]; and Phys. Rev. Lett. **102**, 161301 (2009) [arXiv:0902.3657].
- [2] E.M. Lifshitz, Zh. Eksp. Toer. Fiz. **11**, 255; 269 (1941).
- [3] A. Padilla, J. Phys. Conf. Ser. **259**, 012033 (2010) [arXiv:1009.4074]; T.P. Sotiriou, J. Phys. Conf. Ser. **283**, 012034 (2011) [arXiv:1010.3218]; P. Hořava, Class. Quantum Grav. **28**, 114012 (2011) [arXiv:1101.1081]; T. Clifton, P.G. Ferreira, A. Padilla, and C. Skordis, arXiv:1106.2476.
- [4] S. Mukohyama, Class. Quantum Grav. **27**, 223101 (2010) [arXiv:1007.5199].
- [5] A.I. Vainshtein, Phys. Lett. B **39**, 393 (1972).
- [6] V.A. Rubakov and P.G. Tinyakov, Phys. -Uspekhi, **51**, 759 (2008); K. Hinterbichler, arXiv:1105.3735.
- [7] K. Izumi and S. Mukohyama, Phys. Rev. D **84**, 064025 (2011) [arXiv:1105.0246].
- [8] A.E. Gumrukcuoglu, S. Mukohyama, and A. Wang, arXiv:1109.2609.
- [9] A. Wang and Q. Wu, Phys. Rev. D **83**, 044025 (2011) [arXiv:1009.0268].
- [10] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990); D. H. Lyth, K. A. Malik and M. Sasaki, JCAP **0505**, 004 (2005) [arXiv:astro-ph/0411220].
- [11] P. Hořava and C.M. Melby-Thompson, Phys. Rev. D **82**, 064027 (2010) [arXiv:1007.2410].
- [12] A. Wang and Y. Wu, Phys. Rev. D **83**, 044031 (2011) [arXiv:1009.2089].
- [13] A.M. da Silva, Class. Quantum Grav. **28**, 055011 (2011) [arXiv:1009.4885].
- [14] Y.-Q. Huang and A. Wang, Phys. Rev. D **83**, 104012 (2011) [arXiv:1011.0739] (Paper I).
- [15] K. Lin, A. Wang, Q. Wu, and T. Zhu, Phys. Rev. D **84**, 044051 (2011) [arXiv:1106.1486].
- [16] J. Kluson, Phys. Rev. D **83**, 044049 (2011) [arXiv:1011.1857].
- [17] T. Zhu, Q. Wu, A. Wang, and F.-W. Shu, Phys. Rev. D **84**, 101502 (R) (2011) [arXiv:1108.1237].
- [18] T. Zhu, F.-W. Shu, Q. Wu, and A. Wang, to appear in Phys. Rev. D (2012) [arXiv:1110.5106].
- [19] D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010) [arXiv:0909.3525]; JHEP, **1104**, 018 (2011) [arXiv:1007.3503].
- [20] I. Kimpton and A. Padilla, JHEP, **07**, 014 (2010) [arXiv:1003.5666].
- [21] J. Greenwald, J. Lenells, J. X. Lu, V. H. Satheeshkumar, and A. Wang, Phys. Rev. D **84**, 084040 (2011) [arXiv:1105.4259].
- [22] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1973).
- [23] F.J. Tipler, Nature, **270**, 500 (1977).
- [24] S.A. Hayward, Phys. Rev. D **49**, 6467 (1994); Class. Quantum Grav. **17**, 1749 (2000).
- [25] A. Wang, Phys. Rev. D **68**, 064006 (2003); *ibid.*, **D72**, 108501 (2005); Gen. Relativ. Grav. **37**, 1919 (2005); A.Y. Miguelote, N.A. Tomimura, and A. Wang, *ibid.*, **36**, 1883 (2004); and P. Sharma, A. Tziolas, A. Wang, and Z.-C. Wu, Inter. J. Mod. Phys. A **26**, 273 (2011) [arXiv:0901.2676].
- [26] R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D **80**, 024003 (2009) [arXiv:0904.3670]; S.-S. Kim, T. Kim, and Y. Kim, *ibid.*, **D80**, 124002 (2009); E.O. Colgain and H. Yavartanoo, JHEP, **08**, 021 (2009) [arXiv:0904.4357]; J.-Z. Tang, arXiv:0911.3849; E. Gruss, Class. Quantum Grav. **28**, 085007 (2011); J.M. Romero, J.A. Santiago, O. Gonzalez-Gaxiola, and A. Zamora, Mod. Phys. Lett. A **25**, 3381 (2010).
- [27] S. Maeda, S. Mukohyama and T. Shiromizu, Phys. Rev. D **80**, 123538 (2009) [arXiv:0909.2149].

- [28] T. Sotiriou, M. Visser, and S. Weinfurtner, Phys. Rev. Lett. **102**, 251601 (2009) [arXiv:0904.4464]; JHEP, **10**, 033 (2009) [arXiv:0905.2798].
- [29] E. Kiritsis and G. Kofinas, Nucl. Phys. B **821**, 467 (2009) [arXiv:0904.1334].
- [30] J. Greenwald, A. Papazoglou, and A. Wang, Phys. Rev. D **81**, 084046 (2010) [arXiv:0912.0011].
- [31] J. J. Greenwald, V.H. Satheeshkumar, and A. Wang, JCAP, **12**, 007 (2010) [arXiv:1010.3794].
- [32] J. Alexandre and P. Pasipoularides, Phys. Rev. D **83**, 084030 (2011) [arXiv:1010.3634]; arXiv:1108.1348.
- [33] P. Painleve, C. R. Acad. Sci. (Paris) **173**, 677 (1921); A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**, 1 (1922).
- [34] A. Wang and R. Maartens, Phys. Rev. D **81**, 024009 (2010) [arXiv:0907.1748]; A. Wang, D. Wands, and R. Maartens, JCAP, **03**, 013 (2010) [arXiv:0909.5167]; A. Wang, Phys. Rev. D **82**, 124063 (2010) [arXiv:1008.3637].
- [35] B. Chen, Q.G. Huang, Phys. Lett. B **683**, 108 (2010) [arXiv:0904.4565].
- [36] P. Hořava and C.M. Melby-Thompson, Gen. Relativ. Grav. **43**, 1391 (2011) [arXiv:0909.3841].
- [37] T. Suyama, JHEP, **01**, 093 (2010) [arXiv:0909.4833]; D. Capasso and A.P. Polychronakos, *ibid.*, **02**, 068 (2010) [arXiv:0909.5405]; J. Alexandre, K. Farakos, P. Pasipoularides, and A. Tsapalis, Phys. Rev. D **81**, 045002 (2010); J.M. Romero, V. Cuesta, J.A. Garcia, and J. D. Vergara, *ibid.*, **D81**, 065013 (2010); S.K. Rama, arXiv:0910.0411; L. Sindoni, arXiv:0910.1329.
- [38] R.-G. Cai and A. Wang, Phys. Lett. B **686**, 166 (2010) [arXiv:1001.0155].
- [39] R. D’Inverno, *Introducing Einstein’s Relativity* (Clarendon Press, Oxford, 1992).
- [40] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
- [41] E. Abdalla, and A.M. da Silva, arXiv:1111.2224.
- [42] K. Lin and A. Wang, “*Quasinormal modes in general covariant theory of Horava-Lifshitz gravity*,” in preparation.